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APPLICABILITY OF LINEAR ELASTIC FRACTURE MECHANICS FOR THE TESTING OF PLASTICS

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Abstract

Linear Elastic Fracture Mechanics (LEFM) is based on the Williams–Irwin equations containing a singularity at the crack tip. To avoid the existence of infinite stresses a small plastic zone around the crack tip is assumed, the shape of which should be calculated by inserting the formula for the working stress into the Williams–Irwin equations. The choice of the type of working stress is free, usually the von Mises theory is applied producing the well-known liver-shaped plastic zone. The width of it along the crack plane is considered as an extension of the crack length. The state of stress at the crack tip in the case of plane strain is nearly hydrostatic, especially for values of the Poissons ratio approximating $\nu = 0.5$. As the von Mises criterion neglects the influence of the hydrostatic component of the stress state upon the elastic–plastic transition, this does not seem to be suitable if applied for investigating the fracture-mechanic properties of plastics. Two other hypotheses – which take into account the influence of the hydrostatic part of the stress state – were investigated for the determination of the size and shape of the plastic zone ahead the crack tip. The application of these for some types of plastics considered as 'rigid' revealed , that only thermosetting plastics are really suitable for strict LEFM investigation for producing the stress intensity factor as a material property.

Keywords: fracture mechanics, plastics, working stress.

Linear elastic fracture mechanics (LEFM) is based on the continuum mechanics of elastic bodies applied to specimens containing cracks. As the crack tips perform singularities where the stresses tend to infinite, complete failure of the specimen would be caused by incremental loads. To obtain agreement between the theory and experimental evidence a deviation from linear elastic behaviour in the vicinity of the crack tip has to be supposed. The investigation of the fracture behaviour of materials is usually performed on plane specimens having constant thickness where plane stress distribution can be assumed at their surface and plane strain at their plane of symmetry. For mode I cracking – as to be investigated in the following – the stress distribution in the vicinity of the crack tip is described by the well-known formulas of WILLIAMS and IRWIN [1]. The necessary deviation from linear elasticity is explained by IRWIN and MCCLINTOCK [2] by the existence of a small plastic region around the crack tip. This has a liver-like shape and can be evaluated by inserting the formula for the working stress (as criterion for the elastic-plastic transition) into the Williams–Irwin formulas. The choice of the failure or plasticity criterion on which the working stress is based remains free. Usually the von Mises criterion is applied producing a plastic region the shape of which can be found practically in all textbooks about fracture mechanics *Fig. 1*, showing the transition from plane stress at the surface to plane strain in the interior of the test specimen. As proceeding inward from the surface, this transition develops fast. It is usually considered in applications of materials testing that the actual value for the crack-sensitivity of the material is described well by the state of plane strain and the thin surface layer of plane stress performs only a negligible inaccuracy in determining the stress intensity factor (s.i.f.) As in the region of the plastic strain the stress distribution differs from that outside of it, the evaluation of the s.i.f. has to be performed basing on the elastic region of the specimen. This leads to an extension of the crack length a by a_p as shown in *Fig. 1*, and the s.i.f. can be calculated

$$K_I = \sigma_{y0} \cdot \sqrt{\pi(a + a_{pa})} \tag{1}$$

provided that a_p has to be small against a. In *Eq.* (1) σ_{yo} is the crack opening stress in sufficient distance from the crack tip.

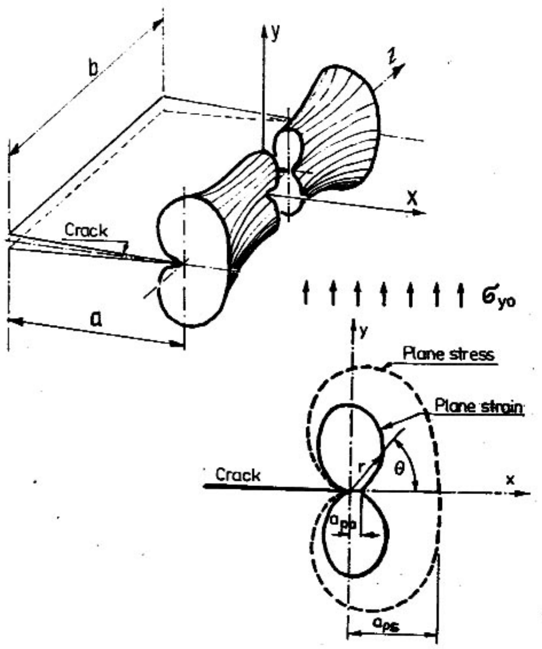

Fig. 1. Upper part: Supposed shape of the plastic zone around the crack tip of a specimen of constant width b containing a crack of the length a loaded according to mode I. fracture test.
Lower part: side view of the specimen showing the size of the plastic zone and the extension of the crack length as suggested by IRWIN and MCCLINTOCK [2].

The two principal stresses in the xy-plane, as derived from the Williams–Irwin formulas, are

$$\begin{aligned}\sigma_1 &= \frac{K_I}{\sqrt{2 \cdot r \cdot \pi}} \cdot \cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\right]\\ \sigma_2 &= \frac{K_I}{\sqrt{2 \cdot r \cdot \pi}} \cdot \cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\right],\end{aligned} \tag{2a}$$

where r and Θ are the polar co-ordinates of an arbitrary point based on the crack tip as shown in the lower part of *Fig. 1*. The third principal stress is in the z-direction and is

$$\sigma_3 = 0 \tag{2b}$$

for plane stress and

$$\sigma_3 = \frac{K_I}{\sqrt{2 \cdot r \cdot \pi}} \cdot 2 \cdot \nu \cdot \cos\frac{\theta}{2} \tag{2c}$$

for plane strain with ν the Poissons ratio.

To obtain the boundary of the non-elastic zone formulas (2a) –(2c) have to be inserted into the equation for the working stress which may be interpreted now as the condition for elastic-plastic transition. Usually the von Mises criterion is applied, which can be written as follows:

$$2 \cdot \sigma_F^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \tag{3}$$

with the yield stress σ_F. Introducing the notation

$$\Phi = \frac{K_I^2}{2 \cdot \pi \cdot \sigma_F^2},$$

the inserting of *Eqs*. (2a) and (2b) into (3), the value of the boundary radius r is obtained. For plane stress as:

$$r_{ps} = \Phi \cdot \cos^2\frac{\theta}{2}\left[4 - 3 \cdot \cos^2\frac{\theta}{2}\right] \tag{4}$$

and for plane strain, applying *Eq*. (2c) instead of (2b):

$$r_{pa} = \Phi \cdot \cos^2\frac{\theta}{2}\left[4\left(1-\nu+\nu^2\right) - 3 \cdot \cos^2\frac{\theta}{2}\right] \tag{5}$$

The distributions r_{ps}/Φ and r_{pa}/Φ for $\nu = 0.36$ and $\nu = 0.5$ are shown in *Fig. 2*.

The extension a_P of the crack length can be obtained – as already mentioned – from Formulas (4) and (5) by inserting $\Theta = 0$. This yields

$$r_{ps(\theta=0)} = a_{ps} = \Phi \tag{6}$$

and

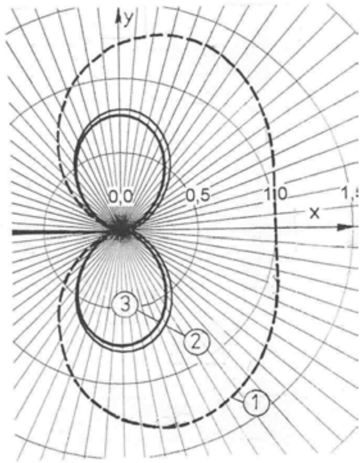

Fig. 2. The size of the plastic zone around the crack tip presented in dimensionless form r/Φ calculated applying the von Mises working stress. 1. Plane stress; 2. Plane strain with $\nu = 0.36$; 3. Plane strain with $\nu = 0.5$.

$$r_{pa(\theta=0)} = a_{pa} = \Phi \cdot \left[4\left(1 - \nu + \nu^2\right) - 3\right]. \tag{7}$$

Ratio a_{pa}/Φ is plotted in *Fig.* 3 over the Poissons ratio ν marking its region for metals and plastics. It shows that a_{pa} becomes very small when approaching $\nu \to 0.5$, indicating compared with *Eqs.* (2a) and (2c) that in the case of plane strain the stress state at the crack tip approaches to be hydrostatic. As the von Mises theory is based on the assumption of depending only on the deviatoric part of the stress state neglecting the hydrostatic part of it, it seems inconvenient in the case of investigations on plastics because of their relative high value of Poissons ratio.

An attempt was made therefore to repeat the foregoing investigation applying such hypotheses for the elastic–plastic transition which are based on both (deviatoric and hydrostatic) parts of the stress state .

Two such hypotheses are known. They are based on the different values of the yield stress σ_{F+} for pure tension and σ_{F-} for pure compression. Their ratio $\Psi = \sigma_{F+}/\sigma_{F-}$ appears in the formulas for these hypotheses as described in [3] and [4]. In the first of them the working stress σ_w depends linearly on the mean stress $3\sigma_k = \sigma_1 + \sigma_2 + \sigma_3$ representing the hydrostatic part of the stress state. It is called 'conical' theory and presented in the Russian literature as the theory of MIROLJUBOV [3]. The formula derived for the working stress according the this

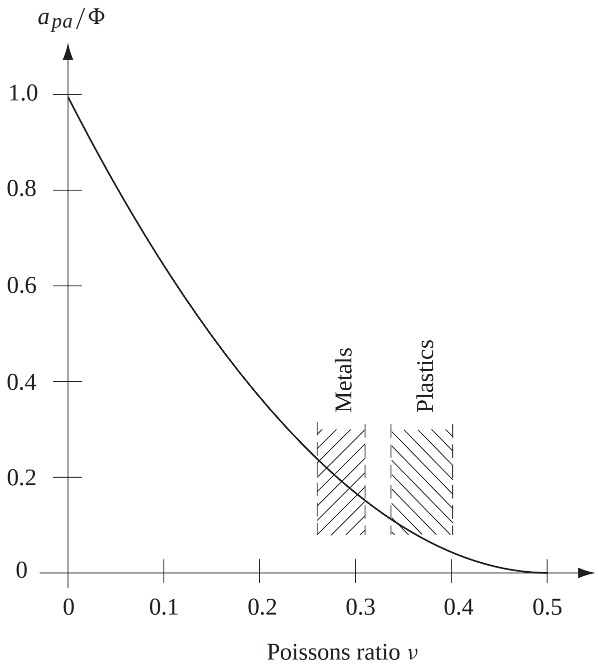

Fig. 3. The suggested extension of the crack length for plane strain over the Poissons ratio, based on the von Mises working stress

theory is

$$\sigma_w = \frac{1-\psi}{2}(\sigma_1+\sigma_2+\sigma_3) + \frac{1+\psi}{2}\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \tag{8}$$

In the second theory, the hydrostatic stress component appears in the second power. It is referred to as the 'parabolic' theory and presented in the Russian literature as the theory of BALANDIN. The formula for the working stress in this case is

$$\sigma_w = \frac{1-\psi}{2}(\sigma_1+\sigma_2+\sigma_3) + \sqrt{\left(\frac{1-\psi}{2}\right)^2 \cdot (\sigma_1+\sigma_2+\sigma_3)^2 + \frac{\psi}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \tag{9}$$

BARDENHEIER [5] checked these formulas experimentally for different plastics and found a fair agreement for their elastic-plastic transition also in the case of considering the theory as failure criterion.

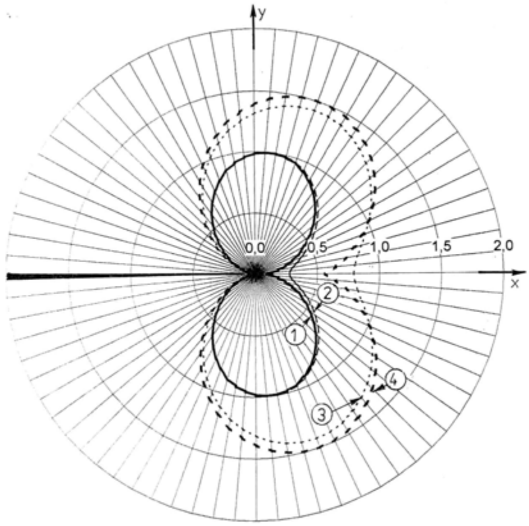

Fig. 4. The shape of the plastic zone around the crack tip in plane strain calculated basing on the 'conical' theory for the working stress. 1. $\Psi = 0.8$, $\nu = 0.36$; 2. $\Psi = 0.8$, $\nu = 0.5$; 3. $\Psi = 0.5$, $\nu = 0.36$; 4. $\Psi = 0.5$, $\nu = 0.5$.

Inserting the formulas (2a) and (2b) for the plane stress as well as (2a) and (2c) for plane strain into formulas (8) and (9) the following formulas could be obtained for the radius of the boundary of the plastic region.

According to the 'conical' theory

plane stress:

$$r_{cs} = \Phi \cdot \cos^2\frac{\theta}{2}\left[1-\psi+\frac{1+\psi}{2}\sqrt{4-3\cos^2\frac{\theta}{2}}\right]^2 \tag{10}$$

plane stress:

$$r_{ca} = \Phi \cdot \cos^2\frac{\theta}{2}\left[(1-\psi)(1+\upsilon)+\frac{1+\psi}{2}\sqrt{4\left(1-\upsilon+\upsilon^2\right)-3\cdot\cos^2\frac{\theta}{2}}\right] \tag{11}$$

and for the 'parabolic' theory

plane stress:

$$r_{ps} = \Phi \cdot \cos^2\frac{\theta}{2}\left[1-\psi+\sqrt{(1+\psi)^2-3\cdot\psi\cdot\cos^2\frac{\theta}{2}}\right]^2 \tag{12}$$

plane strain:

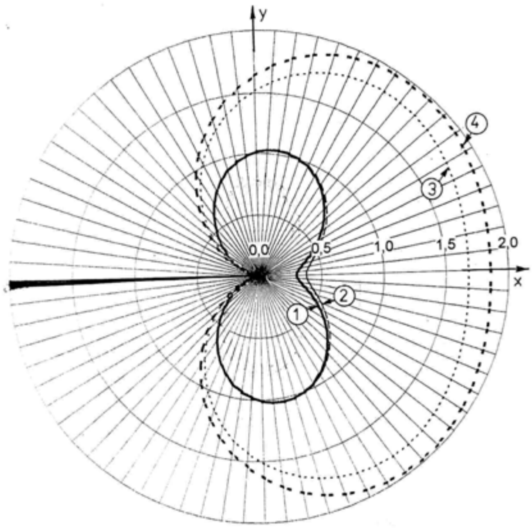

Fig. 5. The shape of the plastic zone in plane strain calculated basing on the parabolic theory. The numbers have the same meaning as in *Fig. 4*.

$$r_{pa} = \Phi \cdot \cos^2 \frac{\theta}{2} \cdot \Bigg[(1-\psi)(1+\nu) + \sqrt{(1-\psi)^2 (1+\nu) + \psi \left\{ 4\left(1-\nu-\nu^2\right) - 3 \cdot \cos^2 \frac{\theta}{2} \right\}} \Bigg] \tag{13}$$

As the state of plane strain is considered to be of interest for materials testing, only the plot of r_{ca}/Φ and r_{pa}/Φ based on formulas (11) and (13) are shown in *Figs. 4* and *5*. The crack extension $a_{pa} = r_{pa(\Theta=0)}$ and $a_{ca} = r_{ca(\Theta=0)}$ are plotted over Ψ in *Fig. 6*. For comparison *Fig. 7* shows the same diagrams for plane stress for both ('conical' and 'parabolical') theories.

To check the validity of the formulas derived, the values of a_{pa} and a_{ca} are calculated for some types of plastics for which the basic strength values could be obtained and considered to be 'rigid'. The results are collected in *Table 1*. The difference between them is obvious. As it can be seen from the table, types 1 – 4, which are thermoplastics, show considerably high values for a_{pa} and a_{ca} from which the highest – polycarbonate – may represent the boundary of applicability of the formulas derived. According to WILLIAMS [6] these plastics investigated by him exhibited a dependence between s.i.f. and velocity of loading, thus revealing a material behaviour differing from that of metals, leaving some doubt about the applicability of LEFM for them. In the contrary the thermosetting type plastics (Nr. 5 and 6 of *Table 1*) show extremely low values of a_{pa} and a_{ca} in good agreement

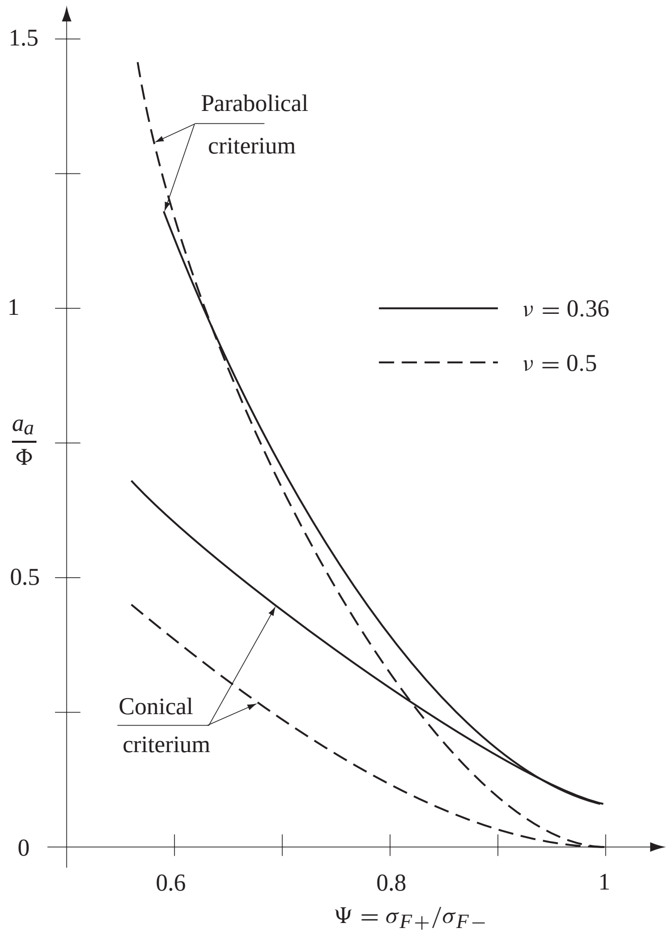

Fig. 6. The extension of the crack length in plane strain over ratio Ψ.

with data presented by BEER [7].

It is not clear whether preference should be given to the ‘conical’ or the ‘parabolical’ theory. Further work is necessary in this direction.

1. Conclusions

The well-known procedure to take into account the local plastic material behaviour of LEFM-specimen in the vicinity of the crack tip by extending the crack length a by an incremental value a_p was investigated for some type of plastics. The usual way to apply the von Mises working stress for the determination of a_p was

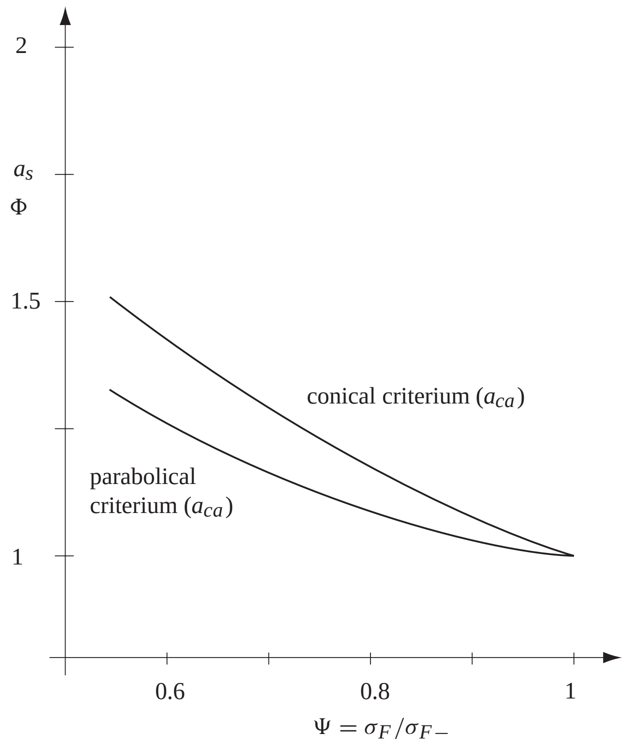

Fig. 7. The same as in *Fig. 6* but for plane stress

Table 1. The size of the plastic zone ahead of the crack tip for different plastics considered as 'rigid' (Approximate values)

Nr.	Type	Ψ	K_I N/mm$^{3/2}$	σ_B MPa	Φ mm	a_{pa} mm	a_{ca} mm
1	PMMA	0.67–0.71	30–45	60	$4–8 \cdot 10^{-2}$	$4.8 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$
2	PS	0.67–0.71	30	30–50	$6–16 \cdot 10^{-2}$	$8.6 \cdot 10^{-2}$	$4.6 \cdot 10^{-2}$
3	PC	0.7	120–160	62–90	0.5–0.6	$4 \cdot 10^{-1}$	$2 \cdot 10^{-1}$
4	Hard PVC	0.75–0.77	30	45	0.07	$3.5 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$
5	UP	0.67–0.71	16	80–90	$5–6 \cdot 10^{-3}$	$3.8 \cdot 10^{-3}$	$2.2 \cdot 10^{-3}$
6	EP	0.7	16	130	$2.5 \cdot 10^{-3}$	$1.75 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$

found to be infavourable because of neglecting the hydrostatic part of the stress state. Two other hypotheses for the working stress were applied therefore, and formulas for the value of a_p were derived showing much higher values than those based on the von Mises criterion. A check of the derived formulas on different types of 'rigid' plastics revealed different behaviour between thermoplastic and thermosetting plastics. Therefore only the latter can be considered to be suitable for real LEFM treatment.

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